Plücker Relations on Schur Functions

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Abstract

We present a set of algebraic relations among Schur functions which are a multi-time generalization of the "discrete Hirota relations" known to hold among the Schur functions of rectangular partitions. We prove the relations as an application of a technique for turning Plücker relations into statements about Schur functions and other objects with similar definitions as determinants. We also give a quantum analog of the relations which incorporates spectral parameters. Our proofs are mostly algebraic, but the relations have a clear combinatorial side, which we discuss.

1 Introduction

Consider the following relationship among the Schur functions s_λ where λ is a rectangular partition:

$$s_{\langle m^\ell \rangle} s_{\langle m^\ell \rangle} = s_{\langle m+1^\ell \rangle} s_{\langle m-1^\ell \rangle} + s_{\langle m^{\ell+1} \rangle} s_{\langle m^{\ell-1} \rangle}. \qquad (1)$$

Here $\langle m^\ell \rangle$ is the partition with ℓ parts each of size m, whose Young diagram is an $\ell \times m$ rectangle. A. N. Kirillov noticed this fact as a relation among the characters of finite-dimensional representations of $\mathfrak{sl}_n$ while studying the Bethe Ansatz for a one-dimensional system called the generalized Heisenberg magnet [Ki].

In later work, Kirillov and Reshetikhin observed that the relations could be viewed as a discrete dynamical system, known to mathematical physics as the discrete Hirota relations [KR]. The initial conditions are the characters of the fundamental representations of $\mathfrak{sl}_n$, and expressing the solutions in terms of the initial conditions is precisely the Jacobi-Trudi formula for $s_{\langle m^\ell \rangle}$.

In this paper, we present the natural extension of this set of relations to Schur functions of arbitrary partitions. The relations are all of the form

$$s_\lambda s_\lambda = s_{\lambda+\omega_\ell} s_{\lambda-\omega_\ell} + \text{ other terms.} \tag{2}$$

Here we borrow notation from Lie theory: if λ is a partition, then we write $\lambda \pm \omega_\ell$ for the partition obtained by adding or removing a column of height ℓ from the Young diagram of λ; this corresponds to taking the highest weight λ and adding or subtracting the fundamental weight ω_ℓ. We have one such relation for every choice of a partition λ and column height ℓ such that λ has a column of height ℓ to begin with (otherwise $\lambda - \omega_\ell$ does not make sense). The various choices of ℓ should be thought of as independent time directions in which we can evolve the dynamical system.

The "other terms" in equation (2) are also products of two Schur functions, and all have coefficients ± 1. The partitions that appear never have more columns or more outside corners than λ does. Thus we get a hierarchy of sets of relations for partitions with up to k corners; when $k = 1$ we are restricted to rectangular partitions, and we recover equation (1).

We prove the relations by reducing them to the Plücker relations among determinants of minors of a certain matrix, whose construction we define in Section 2. The construction applies not only to Schur functions, which we now view as determinants of the Jacobi-Trudi matrices, but to the determinants of any family of matrices with a similar type of definition. We formalize this notion, giving several other examples and a general version of the construction. The Plücker relations themselves are briefly reviewed in Section 3.

In Section 4 we state and prove the relations. We also prove a generalization of the relations to ones which include "shifts" or "spectral parameters." The generalizations of Schur functions that satisfy this version of the equations are the quantum analogs of characters for finite-dimensional representations for $U_q(\widehat{\mathfrak{sl}_n})$, and the generalized version may be related to the representation theory of quantum affine algebras, which is not yet well understood.

Finally, while most of the earlier proofs are algebraic, in Section 5 we offer a combinatorial interpretation for the relations in terms of the Littlewood-Richardson rule, in which the coefficients of ± 1 in the other terms mentioned above arise from an inclusion-exclusion argument. We give a completely bijective proof for equation (1), the rectangular Young diagram version, and we conjecture the existence of bijections with certain properties that would lead to a fully combinatorial proof of equation (2) as well.

2 Generalized Jacobi-Trudi sets

We will describe a scheme for translating the Plücker relations among determinants of minors of a matrix into identities of objects defined by a Jacobi-Trudi style determinantal formula. An instance of this technique was used implicitly in [LWZ] to prove some relations among quantum transfer matrices. Our applications will include Schur functions (characters of representations of SL_n), skew Schur functions, and Schur functions with spectral parameters (quantum characters of $U_q(\widehat{\mathfrak{sl}_n})$). We construct the matrix in this section; the Plücker relations are discussed in Section 3.

The heart of the construction is an operation $(A, B) \to A \square B$, where A and B are $n \times n$ matrices and $A \square B$ is an $(n+1) \times (2n+2)$ matrix. The operation can be depicted graphically as:

$$(3)$$

We will first define the operation for our motivating example, the set of Jacobi-Trudi matrices:

$$\big\{ M_\lambda := (h_{\lambda_i - i + j})_{i,j=1}^n \,|\, n \in \mathbb{Z}_{\geq 0},\ \lambda \text{ a partition with } n \text{ parts} \big\}.$$

If h_k is the kth homogeneous symmetric function (so $h_0 = 1$ and $h_k = 0$ for $k < 0$), then $\det(M_\lambda)$ is the Schur function s_λ. We permit λ to end with zeros, so if λ is a partition with n parts then we can obtain s_λ as the determinant of such an $m \times m$ matrix for any $m \geq n$.

Construction 2.1 *Let λ and ν be partitions with n parts. We define the matrix $M = M_\lambda \square M_\nu$, with $n + 1$ columns indexed by $\{1, \ldots, n+1\}$ and*

$2n + 2$ rows indexed by $\{L, R, 1, \dots, n, 1', \dots, n'\}$, as follows:

$$
\begin{aligned}
M_{Lj} &= \delta(j, 1) \\
M_{Rj} &= (-1)^n \delta(j, n+1) \\
M_{ij} &= h_{\lambda_i - i + j}, \ i = 1, \dots, n \\
M_{i'j} &= h_{\nu_i - i + j - 1}, \ i = 1, \dots, n
\end{aligned}
$$

We adopt the notation $[r_1 r_2 \dots r_k]_M$ for the determinant of the $k \times k$ minor of a $k \times n$ matrix M consisting of rows with indices $r_1, \dots, r_k$; when the choice of M is clear from context the subscript will be dropped. Then for $M = M_\lambda \square M_\mu$, we have $[R12 \dots n] = s_\lambda$ and $[L1' \dots n'] = s_\mu$. (The sign of $M_{R,n+1}$ was chosen for convenience precisely to make this happen.) Plücker relations on M will give us relations among Schur functions.

The construction relies on the following property of the set of Jacobi-Trudi matrices $\{M_\lambda\}$: there is a unique way to fill in the $*$ regions in equation (3) so that any $n + 1$-row minor of $M_\lambda \square M_\mu$ with nonzero determinant is again some M_ν, possibly padded with one or two rows and columns that do not affect the determinant. To give a generalization of the construction, we isolate this property.

Definition 2.2 *Let $\mathcal{M}$ be a set of square matrices. Let $\mathcal{R}_n$ denote the set of n-component vectors that appear as rows in any $n \times n$ matrix $M \in \mathcal{M}$, for each $n \in \mathbb{Z}_+$. We say $\mathcal{M}$ is a* generalized Jacobi-Trudi set *if there exist equivalence relations $\sim_n$ on $\mathcal{R}_n$ such that:*

1. *Any two rows of an $n \times n$ matrix $M \in \mathcal{M}$ are $\sim_n$ related,*

2. *If M is an $n \times n$ matrix with nonzero determinant and all of its rows are pairwise $\sim_n$ related, then there is a matrix $M' \in \mathcal{M}$ with the same rows as M (but possibly permuted).*

Consider the operators d_L and d_R, which respectively drop the left and right components of a row vector.

3. *Take any two rows $r_1, r_2 \in \mathcal{R}_n$ such that $d_L(r_1), d_L(r_2) \in \mathcal{R}_{n-1}$. If $r_1 \sim_n r_2$, then $d_L(r_1) \sim_{n-1} d_L(r_2)$. Furthermore, $d_L(r_1) = d_L(r_2)$ only if $r_1 = r_2$. Thus we can talk about d_L acting on the equivalence classes. Likewise, all this must hold for d_R as well.*

4. *If A and B are two $\sim_n$ classes such that $d_L(A) = d_R(B)$ then there is a unique $\sim_{n+1}$ class C such that $d_R(C) = A$ and $d_L(C) = B$.*

4

Our archetypical generalized Jacobi-Trudi set of matrices, of course, is the set of Jacobi-Trudi matrices M_λ defined above. In this case there is only one conjugacy class for each $\sim_n$, and it consists of all rows of the form $(h_k, h_{k+1}, \ldots, h_{k+n-1})$ for $k + n - 1$ nonnegative. We should verify property (2) from the definition: a matrix with n rows of this form, ordered to have nonincreasing values of k, corresponds to a partition unless some row is repeated or some element on the diagonal is h_j for j negative. In either case the resulting matrix has determinant 0.

Other examples of generalized Jacobi-Trudi sets include:

Example 1 The matrices M_λ with the h_k considered as formal variables, *without* the specialization that $h_0 = 1$ and $h_k = 0$ for $k < 0$. There is still only one $\sim_n$ class for each n, but now every matrix formed of n distinct rows from that class has nonzero determinant. ∎

Example 2 The matrices $M_{\lambda/\mu} := (h_{\lambda_i - \mu_j - i + j})_{i,j=1}^n$. The determinants of these matrices are the skew Schur functions $s_{\lambda/\mu}$ corresponding to skew Young diagrams λ/μ, with $\mu \subset \lambda$ (*i.e.* $\mu_i \le \lambda_i$ for all i). In this case, for each n there are infinitely many $\sim_n$ classes, one for each choice of μ: given a row vector $(h_{a_1}, h_{a_2}, \ldots, h_{a_n})$, it can appear in matrices $M_{\lambda/\mu}$ where $\mu_i - \mu_{i+1} = a_{i+1} - a_i - 1$.

The operator d_L (resp. d_R) takes the $\sim_n$ class associated with μ to the $\sim_{n-1}$ class of μ with μ_1 (resp. μ_{n-1}) removed. (Without loss of generality we assume that $\mu_n = 0$.) ∎

Example 3 The set of matrices $T_\lambda(u + c)$, where λ is a partition, u is a formal variable, and $c \in \mathbb{Z}$ is called the shift. We will take the following as a formal definition:

$$T_\lambda(u) := (t_{\lambda_i - i + j}(u + \lambda_1 - \lambda_i + i + j - n - 1))_{i,j=1}^n$$

where λ has n parts, some of which may be zero. Define $s_\lambda^{(u)} := \det(T_\lambda(u))$. As with Schur functions, the $t_k(u)$ can optionally be specialized to $t_0(u) = 1$, $t_k(u) = 0$ for $k < 0$.

We will treat the $s_\lambda^{(u)}$ as formal symbols, but see the remarks following Theorem 4.4 for comments and references on the mathematical physics origins of the objects. Essentially, $s_\lambda^{(u)}$ can be regarded as quantum analogs of characters of representations of $U_q(\hat{\mathfrak{g}})$. If we send the entry $t_k(u + c)$ to h_k and therefore ignore the shift (this is letting $u \to \infty$ in the mathematical

physics literature) we recover the Jacobi-Trudi matrices M_λ and plain Schur functions s_λ.

To understand the equivalence classes here, note that the rows of any matrix $T_\lambda(u+c)$ are of the form

$$(t_a(u+b), t_{a+1}(u+b+1), \ldots, t_{a+n-1}(u+b+n-1))$$

for some choice of integers a and b. The main diagonal of $T_\lambda(u)$ has entries $t_{\lambda_1}(*), t_{\lambda_2}(*), \ldots, t_{\lambda_n}(*)$, while the anti-diagonal has $t_*(u), t_*(u+\lambda_1 - \lambda_2), \ldots, t_*(u+\lambda_1 - \lambda_n)$. It is therefore easy to see that if the row beginning with $t_a(u+b)$ appears in the matrix $T_\lambda(u+c)$, we must have $a+b = \lambda_1 - n + 1$. Therefore each $\sim_n$ class contains all rows which share a common value $a+b$.

We remark that given a partition λ with n parts and an $\sim_n$ class A, there is a unique integer c such that the rows of $T_\lambda(u+c)$ are in A. ■

Finally, we give a version of Construction 2.1 for any generalized Jacobi-Trudi set of matrices, which we will apply to the examples above.

Construction 2.3 *Let $\mathcal{M}$ be a generalized Jacobi-Trudi set of matrices, and take two $n \times n$ matrices $A, B \in \mathcal{M}$. Let $\widetilde{A}, \widetilde{B}$ denote the $\sim_n$ classes of their respective rows.*

We say A and B are compatible *if $d_L(\widetilde{A}) = d_R(\widetilde{B})$. For compatible A, B we can define the $(n+1) \times (2n+2)$ matrix $A \square B$. Let $\widetilde{C}$ be the $\sim_{n+1}$ class such that $d_R(\widetilde{C}) = \widetilde{A}$ and $d_L(\widetilde{C}) = \widetilde{B}$, whose existence and uniqueness is guaranteed by Definition 2.2. The rows of $A \square B$ are indexed by $\{L, R, 1, \ldots, n, 1', \ldots, n'\}$.*

- *Row L is $(1, 0, \ldots, 0)$,*

- *Row R is $(0, \ldots, 0, (-1)^n)$,*

- *Row i for $i = 1, \ldots, n$ is the (unique) row $r_i \in \widetilde{C}$ such that $d_R(r_i)$ is the ith row of A,*

- *Row i' for $i = 1, \ldots, n$ is the (unique) row $r_{i'} \in \widetilde{C}$ such that $d_L(r_{i'})$ is the ith row of B.*

We point out again that $[R12 \ldots n] = \det(A)$ and $[L1' \ldots n'] = \det(B)$. Of course, when $\mathcal{M} = \{M_\lambda\}$ this reduces to Construction 2.1.

3   Plücker relations

The Plücker relations are the algebraic dependencies among the determinants of the various minors of an arbitrary matrix. We quickly review them, fix our notation, and present a few pertinent examples. For details on the subject, see [S].

The Plücker relations are most naturally stated for the $n \times n$ minors of an $n \times 2n$ matrix. Suppose we have such a matrix, and we index its $2n$ rows by $1, \ldots, n, 1', \ldots, n'$. Pick some integer k, $1 \le k \le n$, and then pick $1 \le r_1 < \cdots < r_k \le n$. The relations state that

$$[12\ldots n][1'2'\ldots n'] = \sum_{1 \le s_1 < \cdots < s_k \le n} \sigma_{RS}([1, 2, \ldots, n][1', 2' \ldots, n'])$$

where σ_{RS} exchanges rows r_i with s_i' for $i = 1, \ldots, k$ before evaluating the determinants. We could define Plücker relations more generally, for the minors of a matrix of any size, but they would be a specialization of the above relations to the case where some row in $[12\ldots n]$ is the same as a row in $[1'2'\ldots n']$, resulting in many zeros on the right hand side of the identity.

Example 4 The 7-term Plücker relation arises from choosing $n = 4$ and $k = 2$. It reads:

$$
\begin{aligned}
[\mathbf{12}34][5678] \quad = \quad & [5634][\mathbf{12}78] + [5734][\mathbf{16}28] + [5834][\mathbf{16}72] + \\
& [6734][5\mathbf{12}8] + [6834][5\mathbf{17}2] + [7834][56\mathbf{12}]
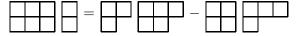\end{aligned}
$$

In this example, we say rows 3 and 4 are *fixed*. ∎

Naturally, we intend to apply the Plücker relations to the generalized Jacobi-Trudi sets of matrices defined in the previous section.

Example 5 Consider the matrix

$$M = \begin{bmatrix} h_3 & h_4 \\ h_2 & h_3 \\ h_1 & h_2 \\ 1 & h_1 \end{bmatrix}$$

Every 2×2 minor of M is a Schur function. Writing the Young diagrams for their Schur functions, the 3-term Plücker relation $[\mathbf{12}][34] = [42][\mathbf{31}] + [32][\mathbf{14}]$ yields

The sign arises from the need to reorder the rows. ∎

Finally, we will be looking at the Plücker relations primarily for the matrices $A\square B$ constructed in Section 2, partitioning rows so that one term of the relation is $\det(A)\det(B)$. To specify an example of this type, we need to choose matrices A and B from a generalized Jacobi-Trudi set, and we need to pick some subset of the rows of either A or B (recall that the $\square$ operation is not symmetric) to be the fixed rows in the identity.

Example 6 Take $\lambda = \langle 2,1,1\rangle$ and $\mu = \langle 4,3,1\rangle$. We will apply the 7-term Plücker relation (Example 4) to $T_\lambda(u)\square T_\mu(u)$ (Example 3). Choosing the first two rows of $T_\mu(u)$ as our fixed rows, we rearrange terms to get:

$$
\begin{aligned}
s^{(u-1)}_{\langle 3,2,1\rangle}s^{(u+1)}_{\langle 3,2,1\rangle} &= s^{(u)}_{\langle 4,3,1\rangle}s^{(u)}_{\langle 2,1,1\rangle} + s^{(u-1)}_{\langle 3,2,2\rangle}s^{(u+1)}_{\langle 3,1,1\rangle} + s^{(u-1)}_{\langle 3,3,3\rangle}s^{(u+3)}_{\langle 1,1,1\rangle}\\
&+ s^{(u)}_{\langle 3,2,2,2\rangle}s^{(u)}_{\langle 3,0\rangle} + s^{(u)}_{\langle 3,3,3,2\rangle}s^{(u+2)}_{\langle 1,0\rangle} - s^{(u)}_{\langle 3,3,3,3\rangle}s^{(u+3)}_{\langle 0,0\rangle}
\end{aligned}
$$

The order of terms was chosen as a precursor to Theorem 4.2. If we ignore the spectral parameters, we get an identity on plain Schur functions:

In the first version, the zero parts of the partitions are necessary if the identity is to work without setting $t_0(u) = 1$, $t_k(u) = 0$ for $k < 0$. If we are willing to make that specialization, we can drop the zero parts, but we must adjust the shifts at the same time: $s^{(u)}_{\langle \lambda_1 \ldots \lambda_n, 0\rangle} = s^{(u-1)}_{\langle \lambda_1 \ldots \lambda_n\rangle}$. In the second version we have already dropped the information about the zero parts. ∎

4   Main Theorem

In this section we present a set of recurrence relations, essentially a discrete dynamical system, to which the Schur functions are a solution. These relations are a generalization of equation (1), a system of relations which hold for the Schur functions of partitions with rectangular Young diagrams. We also present the quantum analog of the relations, in Theorem 4.4 and following comments; this generalizes the relation

$$
s^{(u-1)}_{\langle m^\ell\rangle}s^{(u+1)}_{\langle m^\ell\rangle} = s^{(u)}_{\langle m+1^\ell\rangle}s^{(u)}_{\langle m-1^\ell\rangle} + s^{(u)}_{\langle m^{\ell+1}\rangle}s^{(u)}_{\langle m^{\ell-1}\rangle} \tag{4}
$$

We prove the relations by reducing them to Plücker relations on $M_\lambda \square M_\mu$, defined in Section 2. The simple forms in equations (1) and (4) come from the 3-term Plücker relation. Example 6 in the previous section shows the simplest instance based on the 7-term Plücker relation.

To state the relations, we first need to define some operations on the partition λ, which we associate with its Young diagram $Y = Y(\lambda)$. Let Y be a Young diagram with n outside corners. That is, we take n points $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with $x_1 > \cdots > x_n$ and $y_1 < \cdots < y_n$, and the points in Y are those less than any of the (x_i, y_i) in the product ordering. We identify Y with the partition $\lambda = \langle x_1^{y_1}, x_2^{y_2 - y_1}, \ldots, x_n^{y_n - y_{n-1}} \rangle$. We also say that Y has $n+1$ inside corners, numbered from 0 to n; the ith one has coordinates (x_{i+1}, y_i), where $y_0 = x_{n+1} = 0$.

Definition 4.1 *Let Y be a Young diagram with n outside corners as above, and pick two integers i, j such that $1 \leq i \leq j \leq n$. We define two Young diagrams by the coordinates of their corners:*

$\pi_j^i(Y)$: *take the corners of Y, add 1 to each of $x_{i+1}, \ldots, x_j, y_i, \ldots, y_j$*
$\mu_j^i(Y)$: *take the corners of Y, add -1 to each of $x_{i+1}, \ldots, x_j, y_i, \ldots, y_j$*

These operations respectively add and remove a border strip which reaches from the ith outside corner to the jth inside corner.

We will also want to add or remove several nested border strips. Given integers $1 \leq i_1 < \cdots < i_r \leq j_r < \cdots < j_1 \leq n$, we further define

$$\begin{aligned}
\pi_{j_1 \cdots j_r}^{i_1 \cdots i_r} &= \pi_{j_r}^{i_r} \circ \cdots \circ \pi_{j_1}^{i_1} \\
\mu_{j_1 \cdots j_r}^{i_1 \cdots i_r} &= \mu_{j_r}^{i_r} \circ \cdots \circ \mu_{j_1}^{i_1}
\end{aligned}$$

Thus we add or remove border strips reaching from outside corner i_s to inside corner j_s for $1 \leq s \leq r$.

We apply these definitions of $\pi_{j_1 \cdots j_r}^{i_1 \cdots i_r}$ and $\mu_{j_1 \cdots j_r}^{i_1 \cdots i_r}$ only considering the coordinates of corners, so the various π_j^i and μ_j^i commute. Note that applying π_j^i, for example, might decrease the number of visible corners of Y (by making y_j the same as y_{j+1}), but we ignore this effect in the latter definitions above. Since the intervals $[i_s, j_s]$ are nested, we will never end up with $x_i < x_{i+1}$ or $y_i > y_{i+1}$.

Finally, we borrow notation from Lie theory: given a partition λ, we let $\lambda \pm \omega_\ell$ denote the partition obtained from λ by adding or removing a column of height ℓ to $Y(\lambda)$. If $\lambda = \langle \lambda_1, \ldots, \lambda_m \rangle$ and $\mu = \lambda \pm \omega_\ell$, then $\mu_i = \lambda_i \pm 1$ for $1 \leq i \leq \ell$ and $\mu_i = \lambda_i$ for $i > \ell$. Of course, we cannot take $\lambda - \omega_\ell$ if $\lambda_\ell = \lambda_{\ell+1}$, that is, if $Y(\lambda)$ does not have a column of height ℓ to begin with.

Theorem 4.2 (Main Theorem) *Take a partition λ whose Young diagram $Y(\lambda)$ has n outside corners. Pick an integer k, $1 \le k \le n$, and let ℓ be the kth-shortest column height in $Y(\lambda)$, so $\ell = y_k$ in the coordinates above. Then*

$$s_\lambda s_\lambda = s_{\lambda + \omega_\ell} s_{\lambda - \omega_\ell} + \sum_{r=1}^{\min(k, n-k+1)} \sum_{\substack{1 \le i_1 < \cdots < i_r \le k \\ k \le j_r < \cdots < j_1 \le n}} (-1)^{r-1} s_{\pi_{j_1 \cdots j_r}^{i_1 \cdots i_r}(\lambda)} s_{\mu_{j_1 \cdots j_r}^{i_1 \cdots i_r}(\lambda)}$$

That is, we take a signed double sum over all chains of properly nested intervals $[i_1, j_1] \supset \cdots \supset [i_r, j_r] \ni k$. For each such chain we have the product of two Schur functions, obtained by adding or removing all the corresponding border strips.

Remark 4.3 The recurrence relations can be viewed as defining the multi-time flow of a discrete dynamical system. We think of s_λ as being associated with the lattice point whose ith coordinate is the number of columns in λ of height i. If we allow arbitrary partitions λ, the system is infinite-dimensional; if we restrict ourselves to representations of $\mathfrak{sl}_{n+1}$ it has dimension n.

First, we note that that no partition appearing in Theorem 4.2 has more outside corners than λ does. Second, we observe that the only partition with more columns than λ is $\lambda + \omega_\ell$. Therefore we can solve for $s_{\lambda + \omega_\ell}$ to get a recurrence relation $s_{\lambda + \omega_\ell} = (s_\lambda^2 - \sum \pm s_\pi s_\mu)/s_{\lambda - \omega_\ell}$, expressing $s_{\lambda + \omega_\ell}$ in terms of Schur functions of partition with strictly fewer columns and no more corners. The only initial conditions that need to be specified are for s_λ when λ has no two columns of the same height. ∎

Example 7 Take λ to be the staircase partition $\langle 3, 2, 1 \rangle$ with $n = 3$ corners, and pick $k = 2$. This instance of Theorem 4.2 is the Schur function part of Example 6. The order in which the terms appear there corresponds to taking the double sum over all sets of nested intervals in the order:

$$\underbrace{\{[2,2]\} \quad \{[1,2]\} \quad \{[2,3]\} \quad \{[1,3]\}}_{r=1} \quad \underbrace{\{[1,3] \supset [2,2]\}}_{r=2}$$

We will address the version with spectral parameters in Theorem 4.4. ∎

Proof. The formula is the Plücker relation on $M_{\lambda - \omega_\ell} \Box M_{\lambda + \omega_\ell}$ in which we fix rows $1', \ldots, \ell'$. We index the rows by $\{L, R, 1, \ldots, m, 1', \ldots, m'\}$ as in Construction 2.1, where m is the number of parts of λ. The fixed rows are

therefore those corresponding to rows of $\lambda + \omega_\ell$ which got longer when the column of height ℓ was added.

First we locate the two pieces of Theorem 4.2 outside the double sum. The term $s_{\lambda-\omega_\ell}s_{\lambda+\omega_\ell}$, of course, is the Plücker term $[R12\ldots m][L1'2'\ldots m']$, as we have pointed out several times before. The $s_\lambda s_\lambda$ term is obtained from the Plücker term $[L1\ldots\ell(\ell+1)'\ldots m'][R1'\ldots\ell'(\ell+1)\ldots m]$, in which we swap L with R and every row of $M_{\lambda+\omega_\ell}$ other than the fixed ones with the corresponding row of $M_{\lambda-\omega_\ell}$. This leaves rows $\ell+1$ through m of the two partitions unchanged in length. The exchange of L and R increases by one the lengths of rows 1 through ℓ of $\lambda - \omega_\ell$ and decreases by one the lengths of rows 1' through ℓ' of $\lambda + \omega_\ell$, giving λ in both cases.

To understand the remaining terms, we first observe that if the parts of λ are not all distinct, then the rows of $M_{\lambda-\omega_\ell}\square M_{\lambda+\omega_\ell}$ are not all distinct either: if $\lambda_i = \cdots = \lambda_j$ for some $i < j \le \ell$, then rows $i,\ldots,j-1$ are exactly rows $(i+1)',\ldots,j'$. Similarly, if $\ell < i < j$ then rows $i+1,\ldots,j$ are just rows $i',\ldots,(j-1)'$. The nonzero terms in the Plücker relation are determined by the placement of the non-duplicated rows like i' and j (for $j \le \ell$) or i and j' (for $i > \ell$). Thus we see that the number of corners n, not the length m of the partition, dictates the form of the Plücker relation we get.

Now consider the $[L1\ldots\ell(\ell+1)'\ldots m'][R1'\ldots\ell'(\ell+1)\ldots m]$ term of the relation, already seen to correspond to $s_\lambda s_\lambda$. Observe that we can get any other Plücker term by exchanging some subset of $\{1,\ldots,\ell\}$ from the first determinant with a subset of the same size drawn from $\{R, (\ell+1),\ldots,m\}$ from the second determinant.

Consider the effect of such a switch of a single row, say row a for row b, with $1 \le a \le \ell < b \le m$. As just noted, we may assume $\lambda_a \ne \lambda_{a+1}$ and $\lambda_b \ne \lambda_{b-1}$. Then the effect on the s_λ corresponding to $[R1'\ldots\ell'(\ell+1)\ldots m]$ is as follows: we add an additional part of size λ_a to λ; we remove a part of size λ_b; and we make each of $\lambda_{a+1},\ldots,\lambda_{b-1}$ one larger, corresponding to the newly shifted main diagonal of the matrix M_λ. But this is precisely the description of $\pi_j^i(\lambda)$, where $Y(\lambda)$ has corner coordinates $y_i = a$ and $y_j = b - 1$. Likewise, in the $[L1\ldots\ell(\ell+1)'\ldots m']$ term we change λ into $\mu_j^i(\lambda)$.

One can easily check that the preceding argument still works when our switched row b is instead the one labeled R; in this case the exchange has the effect of π_n^i and μ_n^i. Exchanging subsets larger than a single element is easily seen to mimic the definition of $\pi_{j_1\cdots j_r}^{i_1\cdots i_r}$ and $\mu_{j_1\cdots j_r}^{i_1\cdots i_r}$; the nesting of the intervals arises because the "push" of $Y(\lambda)$ at outside corner a and the "pull" at inside corner b are completely independent.

Finally, we find that after the swap of row a for row b described above,

we always need an odd number of adjacent transpositions to correctly order the rows in the determinants: essentially, we need to exchange rows b and b' but not a and a'. This explains the $(-1)^{r-1}$ term, and completes the proof of Theorem 4.2. ∎

There is a quantum analog of Theorem 4.2 for the Schur functions with spectral parameters defined in Example 3.

Theorem 4.4 *For any partition λ, we can add spectral parameters to the statement of Theorem 4.2 to get*

$$s_\lambda^{(u-1)} s_\lambda^{(u+1)} = s_{\lambda+\omega_\ell}^{(u)} s_{\lambda-\omega_\ell}^{(u)} + \sum \sum \pm s_{\pi(\lambda)}^{(u+*)} s_{\mu(\lambda)}^{(u+*)}$$

where the parameters inside the sum are as follows: given nested intervals $1 \le i_1 < \cdots < j_1 \le n$, set $\alpha = \pi_{j_1 \cdots j_r}^{i_1 \cdots i_r}(\lambda)$ and $\beta = \mu_{j_1 \cdots j_r}^{i_1 \cdots i_r}(\lambda)$. Then the corresponding term in the sum is

$$
\begin{aligned}
s_\alpha^{(u)} \; s_\beta^{(u+\lambda_1-\beta_1)} & \qquad \text{if } j_1 = n \\
s_\alpha^{(u-1)} \; s_\beta^{(u+\lambda_1-\beta_1+1)} & \qquad \text{if } j_1 < n
\end{aligned}
$$

The case when $k = n$ is the subject of [LWZ], where it is proved, as here, by reducing to Plücker relations. Note that for $k = 1$ or n, the double sum is actually a single sum and no negative terms appear.

Proof. As pointed out in Example 3, by appropriate choice of a shift c, we can lift the matrix M_λ to a matrix $T_\lambda(u + c)$ whose rows are in whatever equivalence class we choose. Thus all we will do is pick some equivalence class, lift rows $1, \ldots, m, 1', \ldots, m'$ of $M_{\lambda-\omega_\ell}\square M_{\lambda+\omega_\ell}$ to that class, and read off the necessary shifts for each minor of our matrix to appear in the Plücker relations. Our choice of equivalence class is almost irrelevant; a different choice would just correspond to adding a constant to u in the final relation.

We follow convention by choosing our equivalence class so that we are dealing with minors of the matrix $M_{\lambda-\omega_\ell}(u)\square M_{\lambda+\omega_\ell}(u)$, whose $2m$ rows other than L and R all look like

$$(t_{\lambda_1-c}(u-m+c), t_{\lambda_1+1-c}(u-m+1+c), \ldots, t_{\lambda_1+m-c}(u+c))$$

The row with label $1'$ has this form with $c = 0$, while the row with label 1 has $c = 1$. When we drop the left or right components of these rows, respectively, we get the top rows of the matrices $M_{\lambda+\omega_\ell}(u)$ and $M_{\lambda-\omega_\ell}(u)$, as desired.

Given a minor corresponding to $s_*^{(u+c)}$, to identify the shift c, recall that the top right entry in the matrix is $t_*(u+c)$. Thus we can easily see that the $[R1'\ldots\ell'(\ell+1)\ldots m][L1\ldots\ell(\ell+1)'\ldots m']$ term of the Plücker relation corresponds to $s_\lambda^{(u-1)}s_\lambda^{(u+1)}$, again by looking at the rows 1 and $1'$ examined above.

Using the same reasoning, we see that for any $\alpha = \pi_{j_1\ldots j_r}^{i_1\ldots i_r}(\lambda)$, the associated minor is either $[R1'\ldots]$ (if row R was not swapped away) or $[1'\ldots]$ (if row R was traded). In the first case, we again end up with $s_\alpha^{(u-1)}$; in the second case, we get $s_\alpha^{(u)}$. Row R is swapped if and only if $j_1 = n$, of course: this is the same as saying the partition α has one more part than λ if and only if we added a border strip that reached the bottom row.

Determining the shift of $\beta = \mu_{j_1\ldots j_r}^{i_1\ldots i_r}(\lambda)$ is more difficult because its top row, other than L and possibly R, might be any of $1, 2, \ldots, \ell, \ell+1$. (Indeed, in Example 3, each of these occurs.) To sidestep this difficulty, we note that the top row of the minor giving rise to β begins $t_{\beta_1}(*)$. Assume that row R was not traded. Since we already know the top row must look like $(t_{\lambda_1+1-c}(*), \ldots, t_*(u+c))$, we conclude that $\beta_1 = \lambda_1+1-c$, so $c = \lambda_1-\beta+1$. Likewise, if row R was traded, the top row is one term shorter and ends with $t_*(u+c-1)$, and the shift decreases by one, to $\lambda_1 - \beta_1$. ∎

We conclude this section with a few comments on the relevance of the quantum version of the theorem.

Remark 4.5 When we restrict λ to being a partition with one corner, *i.e.* a rectangle, we are dealing with the 2-dimensional discrete dynamical system

$$Q_{m+1}^\ell(u) = \frac{Q_m^\ell(u-1)Q_m^\ell(u+1) - Q_m^{\ell-1}(u)Q_m^{\ell+1}(u)}{Q_{m-1}^\ell(u)} \tag{5}$$

for $\ell = 1, \ldots, n$ and $m \in \mathbb{Z}_+$. Theorem 4.4 states that this system has a solution in which $Q_m^\ell(u)$ is set to $s_{\langle m^\ell \rangle}^{(u)}$, an object which reduces to $s_{\langle m^\ell \rangle}$ if we ignore the spectral parameter.

The objects $s_\lambda^{(u)}$ themselves have a representation-theoretic interpretation. Using the fact that $U_q(\widehat{\mathfrak{sl}_n})$ is quasitriangular in the category of evaluation modules and highest-weight modules, one can find a commutative subalgebra which is a homomorphic image of the Grothendieck ring. The $s_\lambda^{(u)}$ live in this subalgebra and play the role of characters; see [FR] for details. These notions come originally from mathematical physics and integrable systems, where the objects in question are transfer matrices of Toda lattices; see [KNS],[LWZ],[BR] for more.

In this sense, dropping the spectral parameter corresponds to throwing away some of the structure of the Lie algebra and retaining only the action of the embedded subalgebra $U_q(\mathfrak{sl}_n)$. ■

Remark 4.6 Attempts to generalize this picture to Lie algebras of types other than A_n began in [KR],[KNS]. In these cases, it appears that the characters of $U_q(\hat{\mathfrak{g}})$ do satisfy a generalized version of equation (5). Dropping the spectral parameters, though, no longer gives statements about the fundamental representations of $U_q(\mathfrak{g})$, but about certain non-irreducible representations which do satisfy the discrete Hirota equations. The generalization to Lie algebras of types other than A_n is hampered by the fact that there is currently no character formula for representations of $U_q(\hat{\mathfrak{g}})$. ■

Remark 4.7 Recent work of the author ([K]) has shown a stronger result about the generalized discrete Hirota relations, in an attempt to sidestep the lack of a $U_q(\hat{\mathfrak{g}})$ character formula. For each Lie algebra $\mathfrak{g}$, there is a *unique* solution to the recurrence relations in which Q_m^ℓ is the character of a representation of $U_q(\mathfrak{g})$ all of whose weights lie under $m\omega_\ell$ in the weight lattice. That is, we require that Q_m^ℓ is a sum of irreducible characters whose highest weights lie under $m\omega_\ell$, each occurring with nonnegative integer coefficients. This positivity constraint on all of the infinitely many characters Q_m^ℓ is quite rigid.

Theorem 4.4 is the first step in extending this picture from the rectangular case, in which only a small subset of the characters of $U_q(\widehat{\mathfrak{sl}_n})$ appear, to a full n-dimensional system involving all of the characters. One can hope that a generalization of these new recurrence relations to other Lie algebras may give us information on irreducible characters of $U_q(\hat{\mathfrak{g}})$ for which we do not even have conjectural values. ■

5 Combinatorial Considerations

In this section, we look at the preceding formulas for Schur functions purely combinatorially. We offer a simple combinatorial proof of the rectangle version of the formula, and indicate why we believe that the subtraction that appears in Theorem 4.2 arises from inclusion-exclusion of sets labeled by single intervals.

We will multiply Schur functions using the following reformulation of the Littlewood-Richardson rule, taken from [N], where the technology of crystal bases is used to give an analog for Lie algebras of type B, C, D as well.

Construction 5.1 *We wish to find the multiset S of partitions such that $s_\lambda s_\mu = \sum_{\nu \in S} s_\nu$. To do this, let $\mathrm{SSYT}(\mu)$ be the set of all semi-standard Young tableaux of shape μ. For any tableau $T \in \mathrm{SSYT}(\mu)$, we obtain its column word $cw(T) = i_1 i_2 \ldots i_m$ by reading off the numbers in T, reading each column from top to bottom, beginning with the rightmost column and ending with the leftmost.*

Now we let the number k act on the Young diagram $Y = Y(\lambda)$ by adding one box to the kth row, provided $\lambda_k < \lambda_{k-1}$. If $\lambda_k = \lambda_{k-1}$ then the action is illegal. Denote the resulting Young diagram by $Y \leftarrow k$. Then

$$S = \big\{ (((Y \leftarrow i_1) \leftarrow i_2) \cdots \leftarrow i_m) \mid i_1 i_2 \ldots i_m = cw(T) \big\}$$

where T ranges over all tableaux in $\mathrm{SSYT}(\mu)$ such that each action is legal.

Now we will give a purely bijective proof of the recurrence relation for rectangular Young diagrams. A proof was given in [Ki] which did not mention the 3-term Plücker relation, but which made use of information from Lie theory about the dimensions of associated $\mathfrak{sl}_n$ representations.

Theorem 5.2 $\quad s_{\langle m^\ell \rangle} s_{\langle m^\ell \rangle} = s_{\langle m^{\ell+1} \rangle} s_{\langle m^{\ell-1} \rangle} + s_{\langle m+1^\ell \rangle} s_{\langle m-1^\ell \rangle}$

Proof. Consider a tableau $T \in \mathrm{SSYT}(\langle m^\ell \rangle)$ such that the action of $cw(T)$ on the Young diagram of shape $\langle m^\ell \rangle$, as in Construction 5.1, is legal. We consider two cases, based on whether or not the leftmost column of T consists exactly of the numbers $1, 2, \ldots, \ell$.

If so, consider the tableau T' obtained by removing the leftmost column of T. Observe that $T' \in \mathrm{SSYT}(\langle m-1^\ell \rangle)$, and the action of $cw(T')$ on $Y(\langle m+1^\ell \rangle)$ is legal and yields the same Young diagram as the action of $cw(T)$ on $Y(\langle m^\ell \rangle)$. Furthermore, all elements T' of $\mathrm{SSYT}(\langle m-1^\ell \rangle)$ whose actions are legal arise in this way; we need only note that $cw(T')$ never tries to build on column $m+1$ of $Y(\langle m+1^\ell \rangle)$.

Otherwise, the leftmost column of T contains an entry strictly larger than ℓ, and therefore so does every column, as rows of T are weakly increasing. Now note that in any column of T, the smallest number greater than ℓ that appears must be $\ell + 1$. This is clear for the rightmost column, since $cw(T)$ acts legally on $Y(\langle m^\ell \rangle)$, and can be seen inductively working to the left, again because rows weakly increase. Therefore we can consider the tableau T' obtained by removing the $\ell + 1$ from each column and pushing up all the numbers below it; clearly $T' \in \mathrm{SSYT}(\langle m^{\ell-1} \rangle)$. As in the first case, this operation gives a bijection between T acting legally on $Y(\langle m^\ell \rangle)$ and T' acting legally on $Y(\langle m^{\ell+1} \rangle)$. $\blacksquare$

We are currently unable to provide a generalization of this argument to arbitrary partitions λ, but we strongly believe that one does exist. Based on computational examples, we conjecture the following form for a bijective proof of Theorem 4.2.

Conjecture 5.3 *Let λ be a partition with n outside corners, choose a corner k and corresponding weight ω_ℓ, and retain the notions of Theorem 4.2. Let L be the set of $\mathrm{SSYT}(\lambda)$ acting legally on $Y(\lambda)$.*

1. *The tableaux in $\mathrm{SSYT}(\lambda - \omega_\ell)$ which act legally on $Y(\lambda + \omega_\ell)$ can be put in bijection with a subset A of L.*

2. *There are subsets $B^i_j \subseteq L \setminus A$, for each $1 \le i \le k \le j \le n$, such that B^i_j is in bijection with $\mathrm{SSYT}(\mu^i_j(\lambda))$ acting legally on $Y(\pi^i_j(\lambda))$.*

3. *$L = A \cup \bigcup B^i_j$.*

4. *The intersection $B^{i_1}_{j_1} \cap \cdots \cap B^{i_r}_{j_r}$ is nonempty if and only if we can reorder the terms to get $1 \le i_1 < \cdots < i_r \le k \le j_r < \cdots < j_1 \le n$, and in that case it is in bijection with $\mathrm{SSYT}(\mu^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda))$ acting legally on $Y(\pi^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda))$.*

All of the bijections between $\mathrm{SSYT}(\lambda)$ acting on $Y(\lambda)$ and $\mathrm{SSYT}(\alpha)$ acting on $Y(\beta)$ should respect the Young diagrams produced by the two actions.

The conjecture implies Theorem 4.2, using inclusion-exclusion to take the union $\bigcup B^i_j$. We presently do not know the bijections or even how to identify the sets A, B^i_j in L.

Example 8 Taking the Schur function part of Example 6 once again, the only subtraction that takes place is of the term $s_{\langle 3,3,3,3\rangle} s_{\langle 0,0\rangle}$, corresponding to the nested intervals $[1,3] \supset [2,2]$. There is one tableau (the empty tableau) whose shape is the partition of zero. To verify this instance of the conjecture, we need to check that the Young diagram $Y(\langle 3,3,3,3\rangle)$ appears in the terms corresponding to intervals $[1,3]$ and $[2,2]$ once each.

This does happen: the element of $\mathrm{SSYT}(\langle 3,1,1\rangle)$ whose column word is 44234 acts on $Y(\langle 3,2,2\rangle)$, and the element of $\mathrm{SSYT}(\langle 1,0\rangle)$ whose column word is 4 acts on $Y(\langle 3,3,3,2\rangle)$, both producing $Y(\langle 3,3,3,3\rangle)$. ∎

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